1 Lorentz Transformations and Lorentz Groups

1.1 Lorentz transformations

1.1.1 Introduction

Einstein's starting point: *the speed of light in all inertial systems is constant.* Consider the two coordinate systems **O** and **O**$'$ with **O**$'$ moving along the $+x$ axis with the speed $\underline{v}$:

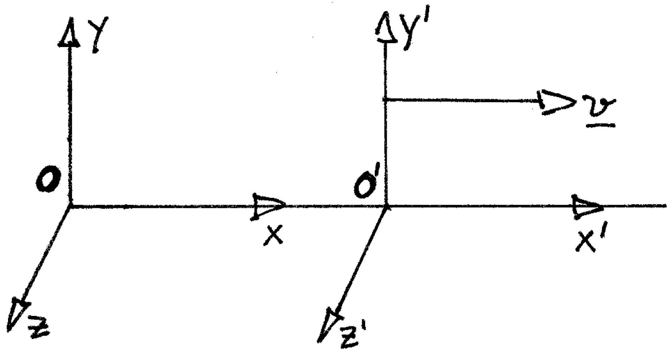

Figure 1.1: The coordinate systems **O** and **O**$'$

Let $x = x'$ for $t = t' = 0$. Then

$$\begin{cases} y' = y \\ z' = z \\ x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}} \\ t' = \dfrac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{cases} \tag{1.1}$$

The inverse transformation is obtained by changing the sign of v:

$$\begin{cases} x = \dfrac{x' + vt}{\sqrt{1 - v^2/c^2}} \\ t = \dfrac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \end{cases} \tag{1.2}$$

Assuming that a light signal from **O** obeys

$$x^2 + y^2 + z^2 = c^2 t^2 \tag{1.3}$$

we find indeed that

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \tag{1.4}$$

The speed of light is the same in **O** and **O**$'$.

— Recall the **Michelson-Morley** experiment (1887).

— Eq. (1.1) was first introduced by **H. A. Lorentz** (1892, 1895). Its use in relativity is due to Einstein.

— Recall **Fitzgerald** (1895): lengths change! (Lorentz: mass changes).

The distance between points 1 and 2:

$$\begin{cases} \Delta x_i = x_i^{(1)} - x_i^{(2)}, \quad i = 1,2,3 \\ \Delta t = t^{(1)} - t^{(2)} \end{cases} \tag{1.5}$$

In the coordinate system **O**:

$$\sum_{i=1}^{3} (\Delta x_i)^2 - c^2 (\Delta t)^2 = 0 \tag{1.6a}$$

In the coordinate system **O′**:

$$\sum_{i=1}^{3} (\Delta x_i')^2 - c^2 (\Delta t')^2 = 0 \tag{1.6b}$$

A coordinate transformation $\mathbf{O} \leftrightarrow \mathbf{O}'$ obeying (1.6) is called a **Lorentz transformation**.

Defining

$$l = ct$$

$$x_4 = \mathrm{i}l = \mathrm{i}ct$$

we can write

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 - \Delta l^2 = 0 \tag{1.6c}$$

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 + \Delta x_4^2 = 0 \tag{1.6d}$$

More generally one can consider the transformation

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 + \Delta x_4^2 = s^2 \tag{1.7}$$

The transformations with $s \neq 0$ lie outside the *"light cone"*.

The general linear transformation becomes[1]

$$x_\mu' = a_\mu + b_{\mu\alpha} x_\alpha \tag{1.8}$$

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[1] From now on Greek indices, $\alpha, \beta, \ldots \mu, \nu$ means the indices run from 1–4, while Latin indices $i, j, \ldots$ run from 1–3. If the same index (for example α) appears twice a summation ($\sum_\alpha$) is to be understood.

1.1.2 Rotations and translations of the four-dimensional space

Theorem : For pure rotations, $a_\mu = 0$, the transformation (1.8) will satisfy (1.7) iff (*if and only if*)[2]

$$b_{\mu\alpha} b_{\nu\alpha} = b_{\alpha\mu} b_{\alpha\nu} = \delta_{\mu\nu} \tag{1.9}$$

Proof: For the the light cone, $s^2 = 0$, (1.6) $\Rightarrow$

$$\left.\begin{aligned} \sum_\mu (x_\mu^{(1)})^2 &= \sum_\mu (x_\mu'^{(1)})^2 \\ \sum_\mu \Delta x_\mu^2 &= \sum_\mu (x_\mu^{(1)} - x_\mu^{(2)})^2 = \sum_\mu (x_\mu'^{(1)} - x_\mu'^{(2)})^2 \end{aligned}\right\} \Rightarrow$$

$$\sum_\mu x_\mu^{(1)} x_\mu^{(2)} = \sum_\mu x_\mu'^{(1)} x_\mu'^{(2)} \tag{1.10}$$

Inserting (1.8) on the RHS (*right hand side*), we get

$$\begin{aligned} \sum_\mu (\sum_\alpha b_{\mu\alpha} x_\alpha^{(1)})(\sum_\beta b_{\mu\beta} x_\beta^{(2)}) &= \sum_{\alpha\beta} (\sum_\mu b_{\mu\alpha} b_{\mu\beta}) x_\alpha^{(1)} x_\beta^{(2)} \\ &\overset{\text{LHS}}{=} \sum_\alpha x_\alpha^{(1)} x_\alpha^{(2)} \end{aligned}$$

This must hold for all $x_\alpha^{(1)}, x_\beta^{(2)} \Rightarrow$

$$\sum_\mu b_{\mu\alpha} b_{\mu\beta} = \delta_{\alpha\beta} \qquad \square$$

As x_1–x_3 are real and x_4 is pure imaginary, all other a_μ and $b_{\mu\alpha}$ are real, except $a_4, b_{41}, b_{42}, b_{43}, b_{14}, b_{24}$ and b_{34} which are pure imaginary.

1.1.3 Lorentz transformations as rotations

For a rotation through ϕ around x_3,

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[2] The δ denotes **Kronecker's** delta

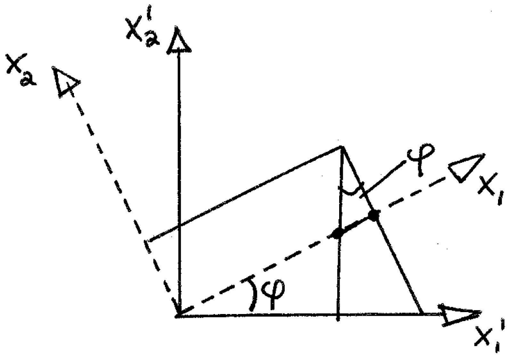

Figure 1.2: Rotation through ϕ

$$\begin{cases} x_1' = x_1\cos\phi - x_2\sin\phi \\ x_2' = x_1\sin\phi + x_2\cos\phi \\ x_3' = x_3 \\ x_4' = x_4 \end{cases} \tag{1.11}$$

Similarly, the Lorentz transformation (1.1) becomes

$$\begin{cases} x_1' = x_1\cos\psi - x_4\sin\psi \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = x_1\sin\psi + x_4\cos\psi \end{cases} \tag{1.12}$$

Now ψ is imaginary, $\sin\psi$ is imaginary and $\cos\psi$ is real. (For real ψ, set $l = ct = -\mathrm{i}x_4, \quad x_4 = \mathrm{i}l$)

$$\begin{cases} x_1' = x_1\cos\psi - \mathrm{i}l\sin\psi \\ \;\, l' = -\mathrm{i}x_1\sin\psi + l\cos\psi \end{cases} \tag{1.13}$$

For the origin $\mathbf{O}'(\mathbf{x_1'} = \ldots = \mathbf{l'} = \mathbf{0})$ we get

$$x_1 = vt = v\frac{l}{c} = \frac{v}{c}l \equiv \beta l$$

Hence (1.13a) $\Rightarrow$

$$x_1 = \mathrm{i}l\frac{\sin\psi}{\cos\psi} = \beta l \Rightarrow$$

$$\beta = \mathrm{i}\tan\psi \tag{1.14}$$

$$\begin{cases} \sin\psi = \dfrac{1}{\sqrt{1+\cot^2\psi}} = \dfrac{1}{\sqrt{1-\beta^{-2}}} = \dfrac{-\mathrm{i}\beta}{\sqrt{1-\beta^2}} \\ \cos\psi = \dfrac{1}{\sqrt{1+\tan^2\psi}} = \dfrac{1}{\sqrt{1-\beta^2}} \end{cases} \tag{1.15}$$

Inserting (1.15) in (1.13), we get

$$\begin{cases} x_1' = \dfrac{x_1 - \mathrm{i}l(-\mathrm{i}\beta)}{\sqrt{1-\beta^2}} = \dfrac{x_1 - vt}{\sqrt{1-\beta^2}} \\ l' = ct' = \dfrac{-\mathrm{i}x_1(-\mathrm{i}\beta) + l}{\sqrt{1-\beta^2}} = \dfrac{l - \beta x_1}{\sqrt{1-\beta^2}} \Rightarrow \\ t' = \dfrac{t - x_1 v/c^2}{\sqrt{1-\beta^2}} \end{cases} \tag{1.16}$$

in agreement with (1.1). *The invariance of* c *yields* (1.1) *!*

1.1.4 Addition of velocities

For two subsequent Lorentz transformations, corresponding to velocitites $\underline{v_1}$ and $\underline{v_2}$, we get

$$\begin{aligned} \beta_{12} &= \mathrm{i}\tan(\psi_1 + \psi_2) = \mathrm{i}\frac{\tan\psi_1 + \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2} \\ &= \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \end{aligned} \tag{1.17}$$

Here $\beta_i = v_i/c$. For $\underline{u} = \underline{u}' + \underline{v}$, we have

$$u = \frac{u' + v}{1 + u'\frac{v}{/}c^2} \tag{1.18}$$

cf. eq. (1.9) in Pyykkö (1975). Classically we would simply have $u = u' + v$, but with (1.18) we see that for $u' \to c, u \to \frac{c+v}{1+v/c} = c$. Even if we would have $v \to c$ at the same time, we get $u \to \frac{c+c}{1+c/c} = c$. The speed of light is not exceeded.

1.1.5 Perpendicular motion

Let now $\mathbf{O}'$ move in the x-direction and a body move in the y-direction:

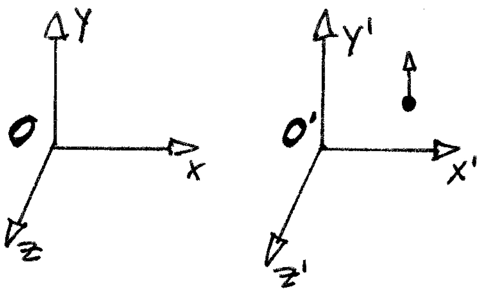

Figure 1.3: **O** and **O'**, again

Consider two different times, t'_1 and t'_2. Then the velocity in the $\mathbf{O}'$ frame becomes

$$u'_y = \frac{\Delta y'}{\Delta t'} = \frac{y'_2 - y'_1}{t'_2 - t'_1} \tag{1.19}$$

With eq. (1.1) we get

$$\begin{cases} y'_2 - y'_1 = y_2 - y_1 \\ t'_2 - t'_1 = \dfrac{t_2 - t_1 - (x_2 - x_1)v/c^2}{\sqrt{1-\beta^2}} = \dfrac{\Delta t - \Delta x (v/c^2)}{\sqrt{1-\beta^2}} \end{cases} \tag{1.20}$$

$$\frac{\Delta y'}{\Delta t'} = \frac{\Delta y \sqrt{1-\beta^2}}{\Delta t - \Delta x (v/c^2)} = \frac{\Delta y}{\Delta t} \frac{\sqrt{1-\beta^2}}{1 - \frac{\Delta x}{\Delta t} v/c^2} \tag{1.21}$$

$$u'_y = \frac{u_y \sqrt{1-\beta^2}}{1 - u_x (v/c^2)} \tag{1.22}$$

By changing the sign of v and primed/unprimed quantities, we can write

$$u_y = \frac{u'_y \sqrt{1-\beta^2}}{1 + u'_x (v/c^2)} \tag{1.23}$$

1.1.6 The relativistic mass transformation

Consider an elastic collision between two bodies having the mass m. Let frame $\mathbf{O}$ have $u_{xA} = 0$ and let $\mathbf{O}'$ be the centre-of-mass frame.

Small letters before impact,
Capital letters after impact

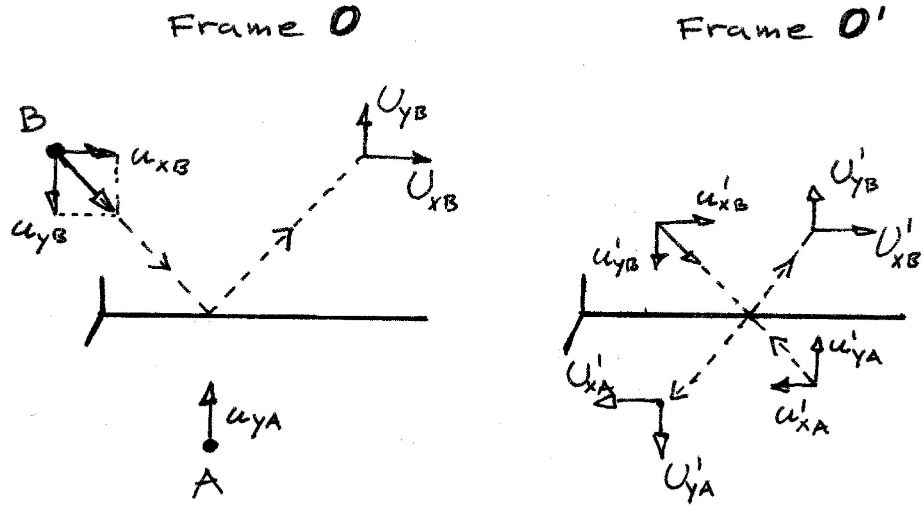

Figure 1.4: $v = u'_{xB} = -u'_{xA}$

From (1.22), in the c.m. coordinates,

$$\begin{cases} u'_{yB} = \dfrac{u_{yB}\sqrt{1-\beta^2}}{1-u_{xB}v/c^2} \\ u'_{yA} = u_{yA}\sqrt{1-\beta^2} \qquad \text{(as } u_{xA}=0\text{)} \end{cases} \tag{1.24}$$

In the c.m. system $|y'_{yA}| = |y'_{yB}| \Rightarrow$

$$u_{yA} = \frac{u_{yB}}{1-u_{xB}v/c^2} \tag{1.25}$$

The laws of nature have to be the same in all inertial frames. Putting the relativistic momentum changes equal in the **O** frame,

$$2m_A u_{yA} = 2m_B u_{yB} \tag{1.26}$$

we get

$$m_B = m_A \frac{u_{yA}}{u_{yB}} = \frac{m_A}{1-u_{xB}v/c^2} \tag{1.27}$$

Note that the two masses are allowed to be different. As

$$v = u'_{xB} = \frac{u_{xB}-v}{1-u_{xB}v/c^2} \Rightarrow \tag{1.28}$$

$v^2 - \frac{2c^2}{u_{xB}}v + v^2 = 0,$

$$v = \frac{c^2}{u_{xB}} (\pm) \sqrt{(\frac{c^2}{u_{xB}})^2 - c^2} \overset{v \leq c}{=} \frac{c^2}{u_{xB}}\left[1-\sqrt{1-(\frac{u_{xB}}{c})^2}\right]. \tag{1.29}$$

Inserting this in (1.27), and letting $u_{yA}, u_{yB} \to 0$,

$$\begin{aligned} m_B &= \frac{m_A}{1-\frac{u_{xB}}{c^2}\frac{c^2}{u_{xB}}\left[1-\sqrt{1-(\frac{u_{xB}}{c})^2}\right]} = \frac{m_A}{\sqrt{1-(\frac{u_{xB}}{c})^2}} \\ &= \frac{m_A}{\sqrt{1-\beta^2}} \end{aligned} \tag{1.30}$$

In three dimensions,

$$\underline{p} = m_0 \frac{\underline{u}}{\sqrt{1-u^2/c^2}} \tag{1.31}$$

According to Einstein, *The Meaning of Relativity*, p. 45, H. A. Lorentz used $\underline{F} = \mathrm{d}\underline{p}/\mathrm{d}t$ with this $\underline{p}$. $\underline{F} \neq m\underline{a}$.

1.1.7 Derivation of $E = mc^2$

The kinetic energy, T, is the work done by a force, F, in accelerating a particle to the speed u:

$$\begin{aligned} T &= \int_{u=0}^{u=u} F\mathrm{d}x = \int_{u=0}^{u=u} \frac{\mathrm{d}}{\mathrm{d}t}(mu)\mathrm{d}x = \int_{u=0}^{u=u} \mathrm{d}(mu)\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \int_{u=0}^{u=u} (m\mathrm{d}u + u\mathrm{d}m)u = \int_{u=0}^{u=u} (mu\mathrm{d}u + u^2\mathrm{d}m) \end{aligned} \tag{1.32}$$

Recalling that

$$\begin{gathered} m = \frac{m_0}{\sqrt{1-u^2/c^2}} \Rightarrow \\ m^2c^2 - m^2u^2 = m_0^2c^2 \end{gathered}$$

and differentiating

$$\begin{aligned} 2mc^2\mathrm{d}m - m^2 \cdot 2u\mathrm{d}u - u^2 \cdot 2m\mathrm{d}m = 0 \quad &\Big| : 2m \\ c^2\mathrm{d}m = mu\mathrm{d}u + u^2\mathrm{d}m \end{aligned} \tag{1.33}$$

which is the integrand (1.32)! $\Rightarrow$

$$T = \int_{u=0}^{u=u} c^2\mathrm{d}m = c^2\int_{m=m_0}^{m=m} \mathrm{d}m = mc^2 - m_0c^2 \tag{1.34}$$

Equivalently,

$$T = m_0c^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right] \tag{1.35}$$

Denoting the total energy mc^2 by E,

$$E = m_0c^2 + T \tag{1.36}$$

where m_0c^2 is the *rest energy*

— **Classical mechanics:** The energy is defined apart from a constant.

— **Relativistic mechanics:** This constant is m_0c^2.

1.1.8 Connection between T and p

From (1.35),

$$\begin{aligned} (T + m_0c^2)^2 &= \tfrac{(m_0c^2)^2}{1-u^2/c^2} \Rightarrow \\ 1 - \frac{u^2}{c^2} &= \tfrac{(m_0c^2)^2}{(T+m_oc^2)^2} \Rightarrow \end{aligned} \tag{1.37}$$

$$u = c\sqrt{1-\frac{(m_0c^2)^2}{(T+m_0c^2)^2}} \tag{1.38}$$

$\Rightarrow$

$$\begin{aligned} p &= \tfrac{m_0u}{\sqrt{1-u^2/c^2}} = \tfrac{m_0}{\sqrt{1-u^2/c^2}}c\sqrt{1-\tfrac{(m_0c^2)^2}{(T+m_0c^2)^2}} \\ &\overset{(1.37)}{=} m_0c\sqrt{1-\tfrac{(m_0c^2)^2}{(T+m_0c^2)^2}} \times \tfrac{T+m_0c^2}{m_0c^2} \\ &= \tfrac{1}{c}\sqrt{(T+m_oc^2)^2-(m_0c^2)^2} \Rightarrow \end{aligned} \tag{1.39}$$

$$\begin{aligned} p^2c^2 &= (T+m_0c^2)^2-(m_0c^2)^2 \Rightarrow \\ E^2 &= (T+mo_c^2)^2 = p^2c^2+m_0^2c^4 \end{aligned} \tag{1.40}$$

1.2 Lorentz matrices

Consider the vector

$$\underline{x_\mu} = (x_1, x_2, x_3, x_4) \tag{1.41}$$

with $x_1 = x, x_2 = y, x_3 = z, x_4 = \mathrm{i}x_0 = \mathrm{i}ct$. Introducing

$$\beta = \frac{v}{c},\ \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{1.42}$$

we can write the transformation (1.16) as

$$\begin{cases} x_1' = \dfrac{x-vt}{\sqrt{1-\beta^2}} & = \gamma(x_1+\mathrm{i}\beta x_4) \\ x_2' = x_2 & \\ x_3' = x_3 & \\ x_4' = \mathrm{i}ct' = \mathrm{i}c\dfrac{t-xv/c^2}{\sqrt{1-\beta^2}} & = \gamma(x_4-\mathrm{i}\beta x_1) \end{cases} \tag{1.43}$$

Using the Einstein summation notation,

$$x_\mu' = a_{\mu\nu}x_\nu \tag{1.44}$$

we have

$$a_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & \mathrm{i}\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i}\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{1.45}$$

We observe that

a. **The length of the vector remains invariant:**

$$
\begin{aligned}
x'_\mu x'_\mu &= \gamma^2(x_1 + \mathrm{i}\beta x_4)^2 + x_2^2 + x_3^2 + \gamma^2(x_4 - \mathrm{i}\beta x_1)^2 \\
&= x_1^2 \underbrace{\gamma^2(1-\beta^2)}_{1} + x^2 + x_3^2 + x_4^2\gamma^2(1-\beta^2) \\
&= x_\mu x_\mu
\end{aligned}
\tag{1.46}
$$

b. **The reality properties of the components are preserved:** x_1, x_2, x_3 are real and x_4 is imaginary.

Definition: $a_{\mu\nu}$ is a **homogenous Lorentz transformation** if

$$
x'_\mu = a_{\mu\nu} x_\nu \tag{1.47}
$$

leaves $x_\mu x_\mu$ invariant and preserves the reality properties of x_μ.

We saw on page 3 that $a_{\mu\nu}$ is orthogonal,

$$
\begin{aligned}
a_{\nu\mu} a_{\lambda\mu} &= \delta_{\nu\lambda} \\
a_{\mu\nu} a_{\mu\lambda} &= \delta_{\nu\lambda}
\end{aligned}
\tag{1.48}
$$

its rows and columns are orthogonal.

Corollary:

$$
\det a = \pm 1 \tag{1.49}
$$

Example 1: Rotations in the 3-D space are Lorentz transformations:

$$
x'_k = a_{kl} x_l, \quad x'_4 = x_4 \tag{1.50}
$$

Here $x_k x_k$ remains invariant (rotation!).

Definition: Lorentz transformations with $a_{44} > 0$ are called *orthochronous*. Orthochronous Lorentz transformations with

$$
\det a = +1 \tag{1.51}
$$

are called *proper* Lorentz transformations

Example 2: For the particular case (1.45),

$$\begin{aligned}
\det a = \det &\begin{vmatrix} \gamma & 0 & 0 & \mathrm{i}\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i}\gamma\beta & 0 & 0 & \gamma \end{vmatrix} \\
&= \gamma \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{vmatrix} - \mathrm{i}\gamma\beta \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{i}\gamma\beta & 0 & 0 \end{vmatrix} = \gamma^2 + (\mathrm{i}\gamma\beta)^2 \\
&= \gamma^2(1-\beta^2) = 1
\end{aligned} \tag{1.52}$$

$a_{44} = \gamma > 1 \Rightarrow a$ is a proper LT.

Example 3: The *inversion*

$$a_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{1.53}$$

is an orthochronous but *improper* LT: $a_{44} > 0, \quad \det a = -1$.

1.3 Infinitesimal Lorentz transformations

Consider the transformations

$$x'_\mu = a_{\mu\nu} x_\nu, \quad a_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu} \tag{1.54}$$

where the $\epsilon_{\mu\nu}$ are infinitesimal quantities. The orthogonality requirement (1.48) $\Rightarrow$

$$(\delta_{\mu\nu} + \epsilon_{\mu\nu})(\delta_{\mu\lambda} + \epsilon_{\mu\lambda}) = \delta_{\nu\lambda} \tag{1.55}$$

Writing out the above multiplication, we get:

$$\underbrace{\delta_{\mu\nu}\delta_{\mu\lambda}}_{\delta_{\nu\lambda}} + \underbrace{\delta_{\mu\nu}\epsilon_{\mu\lambda}}_{\epsilon_{\nu\lambda}} + \underbrace{\epsilon_{\mu\nu}\delta_{\mu\lambda}}_{\epsilon_{\lambda\nu}} + \underbrace{\epsilon_{\mu\nu}\epsilon_{\mu\lambda}}_{\approx 0} = \delta_{\nu\lambda}$$

$$\Rightarrow \epsilon_{\nu\lambda} = -\epsilon_{\lambda\nu} \tag{1.56}$$

where we have assumed the product of two infinitesimals to be negligible.

The reality properties demand that

$$\epsilon^*_{ik} = \epsilon_{ik}, \quad \epsilon^*_{44} = \epsilon_{44}, \quad \epsilon^*_{i4} = -\epsilon_{i4}, \quad \epsilon^2_{4k} = -\epsilon_{4k} \tag{1.57}$$

Hence the ϵ matrix becomes

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 & \mathrm{i}\lambda_1 \\ -\omega_3 & 0 & \omega_1 & \mathrm{i}\lambda_2 \\ \omega_2 & -\omega_1 & 0 & \mathrm{i}\lambda_3 \\ -\mathrm{i}\lambda_1 & -\mathrm{i}\lambda_2 & -\mathrm{i}\lambda_3 & 0 \end{pmatrix} \tag{1.58}$$

The vectors

$$\begin{aligned} \underline{\omega} &= (\omega_1, \omega_2, \omega_3) \\ \underline{\lambda} &= (\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{1.59}$$

are real and infinitesimal but otherwise arbitrary.

Example 4: Let $\underline{\omega} = (\omega_1, \omega_2, \omega_3), \quad \underline{\lambda} = (0, 0, 0) \Rightarrow$

$$a_{\mu\nu} = \begin{pmatrix} 1 & \omega_3 & -\omega_2 & 0 \\ -\omega_3 & 1 & \omega_1 & 0 \\ \omega_2 & -\omega_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.60}$$

This is a three-dimensional (3-D) rotation where $\omega_1, \omega_2, \omega_3$ are the rotation angles about the three axes.

Example 5: Let $\underline{\omega} = (0, 0, 0), \quad \underline{\lambda} = (\lambda_1, 0, 0) \Rightarrow$

$$a_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \mathrm{i}\lambda_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i}\lambda_1 & 0 & 0 & 1 \end{pmatrix} \tag{1.61}$$

Suppose that $v_1/c \ll 1$. Then, recalling (1.45), for

$$a_{14} = \mathrm{i}\gamma\beta$$

$$\mathrm{i}\gamma\beta = \mathrm{i}\frac{v/c}{\sqrt{1 - v^2/c^2}} \approx \mathrm{i}\frac{v_1}{c} + \mathbf{O}((\frac{v_1}{c})^3) \tag{1.62}$$

Thus (1.61) is of the form (1.45), we have a translational Lorentz transformation in the x-direction.

The *generators* of infinitesimal Lorentz transformations are the six matrices

$$\Omega_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.63}$$

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.64}$$

Then we can express

$$\epsilon_{\mu\nu} = \underline{\omega} \cdot \underline{\Omega} + \mathrm{i}\underline{\lambda} \cdot \underline{\Lambda} \tag{1.65}$$

Theorem : The commutators

$$[\Omega_i, \Omega_k] \equiv \Omega_i\Omega_k - \Omega_k\Omega_i, \quad \text{etc.} \tag{1.66}$$

obey

$$\begin{aligned} [\Omega_i, \Omega_k] &= -\delta_{ikl}\Omega_l \\ [\Lambda_i, \Lambda_k] &= -\delta_{ikl}\Omega_l \\ [\Lambda_i, \Omega_k] &= -\delta_{ikl}\Lambda_l \end{aligned} \tag{1.67}$$

where the *permutation symbol*

$$\delta_{ikl} = \begin{cases} +1, & \text{if } i,k,l \text{ all different, } (ikl) \text{ even permutation} \\ -1, & \text{if } i,k,l \text{ all different, } (ikl) \text{ odd permutation} \\ 0, & \text{otherwise} \end{cases} \tag{1.68}$$

Proof: Explicit calculation. E.g.

$$\Omega_1\Omega_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Omega_2\Omega_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[\Omega_1, \Omega_2] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\Omega_3, \quad (123) \text{ even} \qquad \square$$

In a more concise form we can express (1.67) for the six generators

$$I_s = (\Omega_1, \Omega_2, \Omega_3, \Lambda_1, \Lambda_2, \Lambda_3) \tag{1.69}$$

as

$$[I_r, I_s] = C^t_{rs} I_t \tag{1.70}$$

where the C^t_{rs} are called *structure constants*.

1.4 The Lorentz group

The six generators $\Omega_1, \Omega_2, \Omega_3, \Lambda_1, \Lambda_2, \Lambda_3$ generate the **proper Lorentz group**, a *continuous group*, whose elements are

$$\beta_s = (\omega_1, \omega_2, \omega_3, \mathrm{i}\lambda_1, \mathrm{i}\lambda_2, \mathrm{i}\lambda_3) \tag{1.71}$$

The 3D rotations (1.60) form a subgroup. Adding the inversion

$$a_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{1.72}$$

we get the **orthocronous Lorentz group**. Adding to it the *time inversion*

$$a_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{1.73}$$

we get the **full Lorentz group** (with eight generators in all).