2 Relativistic Fields

2.1 Definition

A wave function spanning a representation of the proper Lorentz group is called a *relativistic field*. The only thing required is that

$$[I_r, I_s] = C^t_{rs} I_t \tag{1.70}$$

must hold

In other words, $\psi_\alpha(x)$ is a relativistic field if it transforms according to

$$\psi'_\alpha(x') = [1 + \underline{\omega} \cdot \underline{\Omega} + \mathrm{i}\underline{\lambda} \cdot \underline{\Lambda}]_{\alpha\beta} \psi_\beta(x) \tag{2.1}$$

For pure (3D) rotations,

$$\psi'(x') = \underbrace{[1 + \underline{\omega} \cdot \underline{\Omega}]}_{n\times n \text{ matrix}} \underbrace{\psi(x)}_{n\text{D vector}} \tag{2.2}$$

The value of n is so far open.

On the other hand, the wave function for a particle with spin $\underline{S}$ transforms under a rotation $\underline{\omega}$ of the coordinate axes as

$$\psi'(x') = [1 + \frac{\mathrm{i}}{\hbar} \underline{\omega} \cdot \underline{S}] \psi(x) \tag{2.3}$$

Hence the spin angular momentum

$$\underline{S} = -\mathrm{i}\hbar \underline{\Omega} \tag{2.4}$$

for whatever $\underline{\Omega}$ is chosen.

A wave equation transforming according to a given representation of the Lorentz group describes a particle with a given spin.

2.2 Scalar fields

For one-component wave functions, the $\underline{\Omega}$ and $\underline{\Lambda}$ are 1×1 matrices. Eq. (1.70) $\Rightarrow \Omega_1\Omega_2 - \Omega_2\Omega_1 = 0 \Rightarrow \Omega_3 = 0$, cycl. Thus

$$\underline{\Omega} = \underline{\Lambda} = 0 \tag{2.5}$$

(2.1) $\Rightarrow$

$$\psi'(x') = \psi(x) \tag{2.6}$$

(2.4) $\Rightarrow$

$$\underline{S} = 0 \tag{2.7}$$

Thus a scalar (one-component) field describes a spin-zero particle.

2.3 An $\underline{S} = 1$ field

Consider a four-component wave function A, transforming as

$$A'_\mu(x') = [1 + \underline{\omega} \cdot \underline{\Omega} + \mathrm{i}\underline{\lambda} \cdot \underline{\Lambda}]_{\mu\nu} A_\nu(x) \tag{2.8}$$

with the 4×4 representation in (1.63). Eq. (2.4)⇒

$$\begin{aligned}
\underline{S} &= -\hbar^2 \left[\begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \right] \\
&= 2\hbar^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = 2\hbar^2 \underline{\underline{\mathbf{1}}}
\end{aligned} \tag{2.9}$$

$$S(S+1) = 2 \Rightarrow S = 1 \tag{2.10}$$

$S_z = -1, 0, 1$. This is the *self-representation* of the proper Lorentz group.

2.4 Two-component spinor fields

Can we find six 2×2 matrices, satisfying (1.67)? Try the Pauli matrices $\underline{\sigma}$, which satisfy

$$[\sigma_i, \sigma_k] = 2\mathrm{i}\delta_{ikl}\sigma_l \tag{2.11}$$

An explicit representation is

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.12}$$

Choose

$$\underline{\Omega} = \frac{\mathrm{i}}{2}\underline{\sigma} \tag{2.13}$$

$$\underline{\Lambda} = \pm\frac{\mathrm{i}}{2}\underline{\sigma} \tag{2.14}$$

We can verify that (1.67) holds

$$\begin{cases}
[\Omega_i, \Omega_k] = -\dfrac{1}{4}[\sigma_i, \sigma_k] = -\dfrac{\mathrm{i}}{2}\delta_{ikl}\sigma_l = -\delta_{ikl}\Omega_l \\
[\Lambda_i, \Lambda_k] = -\dfrac{1}{4}[\sigma_i, \sigma_k] = -\dfrac{\mathrm{i}}{2}\delta_{ikl}\sigma_l = -\delta_{ikl}\Omega_l \\
[\Lambda_i, \Omega_k] = \quad \mp [\sigma_i, \sigma_k] = \mp\dfrac{\mathrm{i}}{2}\delta_{ikl}\sigma_l = -\delta_{ikl}\Lambda_l, \quad \text{indeed.}
\end{cases} \tag{2.15}$$

Eq. (2.4) ⇒

$$\underline{S} = -\mathrm{i}\hbar\underline{\Omega} = \frac{\hbar}{2}\underline{\sigma} \tag{2.16}$$

$$\underline{S}^2 = \frac{\hbar^2}{4}\Big[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \Big] = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S(S+1) = \frac{3}{4} \Rightarrow S = \frac{1}{2} \tag{2.17}$$

Consider then the inversion

$$P = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad [1.70]$$

For the matrices ER1.62–(1.64)

$$[P, \Omega] = 0 \tag{2.18}$$

$$\{P, \Lambda\} \equiv P\Lambda + \Lambda P = 0 \tag{2.19}$$

Example 1:

$$\left.\begin{aligned} P\Omega_1 &= \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix} = \begin{pmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{pmatrix} \\ \Omega_1 P &= \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{pmatrix} \end{aligned}\right\} \Rightarrow [P, \Omega_1] = 0$$

$$\left.\begin{aligned} P\Lambda_1 &= \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ -1 & & & \end{pmatrix} = \begin{pmatrix} & & & -1 \\ & & & \\ & & & \\ -1 & & & \end{pmatrix} \\ \Lambda_1 P &= \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \end{aligned}\right\} \Rightarrow \{P, \Lambda_1\} = 0 \quad \square$$

Now , for the two-component case, either $\underline{\Omega} = \underline{\Lambda}$ or $\underline{\Omega} = -\underline{\Lambda}$ (eq. (2.14)) and we cannot satisfy simultaneously both (2.18) and (2.19). **Therefore a two-component $S = \frac{1}{2}$ field cannot be found**.

2.5 Four-component spinor fields

Consider now a four-component function transforming as

$$\psi'(x') = [1 + \underline{\omega} \cdot \underline{\Omega} + \mathrm{i}\underline{\lambda} \cdot \underline{\Lambda}]\psi(x) \tag{2.20}$$

with

$$\underline{\Omega} = \begin{pmatrix} \frac{\mathrm{i}}{2}\underline{\sigma} & \\ & \frac{\mathrm{i}}{2}\underline{\sigma} \end{pmatrix}, \tag{2.21}$$

$$\underline{\Lambda} = \begin{pmatrix} \frac{\mathrm{i}}{2}\underline{\sigma} & \\ & -\frac{\mathrm{i}}{2}\underline{\sigma} \end{pmatrix}. \tag{2.22}$$

These $\underline{\Omega}$ and $\underline{\Lambda}$ satisfy (1.67).

The corresponding angular-momentum operator

$$\underline{S} = -\mathrm{i}\hbar\underline{\Omega} = \frac{\hbar}{2}\begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} \equiv \frac{\hbar}{2}\underline{\sigma}' \tag{2.23}$$

$$\underline{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow S = \frac{1}{2} \tag{2.24}$$

Introduce the four new matrices

$$\underline{\gamma} = \begin{pmatrix} 0 & \mathrm{i}\underline{\sigma} \\ -\mathrm{i}\underline{\sigma} & 0 \end{pmatrix} \tag{2.25}$$

$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.26}$$

These matrices are *Hermitian*,

$$\gamma_\mu^\dagger = \gamma_\mu \tag{2.27}$$

and their anti-commutators satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{2.28}$$

Example 2:

$$\{\gamma_4, \gamma_4\} = 2\begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}^2 = 2\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \text{OK}$$

$$
\begin{aligned}
-\{\gamma_1,\gamma_2\} &= \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & -1 & & \\ -1 & 0 & & \end{pmatrix} \begin{pmatrix} & & 0 & -\mathrm{i} \\ & & \mathrm{i} & 0 \\ 0 & \mathrm{i} & & \\ -\mathrm{i} & 0 & & \end{pmatrix} + \begin{pmatrix} & & 0 & -\mathrm{i} \\ & & \mathrm{i} & 0 \\ 0 & \mathrm{i} & & \\ -\mathrm{i} & 0 & & \end{pmatrix} \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & -1 & & \\ -1 & 0 & & \end{pmatrix} \\
&= \begin{pmatrix} -\mathrm{i} & 0 & & \\ 0 & \mathrm{i} & & \\ & & -\mathrm{i} & 0 \\ & & 0 & \mathrm{i} \end{pmatrix} + \begin{pmatrix} \mathrm{i} & 0 & & \\ 0 & -\mathrm{i} & & \\ & & \mathrm{i} & 0 \\ & & 0 & -\mathrm{i} \end{pmatrix} = 0, \quad \text{OK}
\end{aligned}
$$

The present $\underline{\Lambda}$ and $\underline{\Omega}$ in (2.21–2.22) can be expressed as

$$\underline{\Omega} = \frac{1}{2}(\gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2) \tag{2.29}$$

$$\underline{\Lambda} = \frac{1}{2}(\gamma_1\gamma_4, \gamma_2\gamma_4, \gamma_3\gamma_4). \tag{2.30}$$

In this representation the space inversion is

$$P = \gamma_4. \tag{2.31}$$

This $\underline{\Omega}$, (2.29), commutes with P,

$$[P, \underline{\Omega}] = 0. \tag{2.32}$$

Example 3:

$$[P, \Omega_1] = \frac{1}{2}(\underbrace{\underbrace{\gamma_4\gamma_2\gamma_3}_{-\gamma_2\gamma_4\gamma_3}}_{+\gamma_2\gamma_3\gamma_4} - \gamma_2\gamma_3\gamma_4) = 0.$$

This $\underline{\Lambda}$ anticommutes with P,

$$\{P, \underline{\Lambda}\} = 0 \tag{2.33}$$

Thus the states in this representation can have a well defined parity. The counterparts of our later Dirac matrices are

$$\underline{\alpha} = \mathrm{i}\gamma_4\underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \tag{2.34}$$

$$\beta = \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.35}$$

The γ_μ matrices obey an Algebra.